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Chiral Hierarchies, Compositeness and the Renormalization Group

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Abstract

A wide class of models involve the fine-tuning of significant hierarchies between a strong-coupling "compositeness" scale, and a low energy dynamical symmetry breaking scale. We examine the issue of whether such hierarchies are generally endangered by Coleman-Weinberg instabilities. A careful study using perturbative two-loop renormalization group methods finds that consistent large hierarchies are not generally disallowed.

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I. Introduction

Chivukula, Golden and Simmons [1] have recently examined the question of when it is possible to tune a large hierarchy in a chiral theory with essentially composite scalar bosons. This issue can arise in models such as heavy-quark condensation models [2]–[9], models of broken technicolor [10] or strong extended technicolor [11]–[13]. In ref. [1], the authors argue that such a hierarchy between the compositeness scale Λ and the chiral symmetry breaking scale v is generally endangered by the Coleman–Weinberg phenomenon [14]. Quantum fluctuations drive the chiral symmetry breaking phase transition to be *first* order. This transition must effectively be *second* order if a large hierarchy can exist by fine-tuning. A notable exception to the general result of ref. [1] is the single composite electroweak $I = \frac{1}{2}$ Higgs boson in the top quark condensate models [4], since there is only one quartic coupling constant in the minimal version.

This issue goes beyond the statement that large hierarchies are unnatural because of the fine-tuning of additive quadratically divergent terms. Even if one can remedy that problem, the authors of [1] argue that such a fine-tuning is problematic if the compositeness conditions imply that some of the coupling constants diverge at the scale Λ . If these conclusions are true then the idea of compositeness and the existence of a large gauge hierarchy, $v/\Lambda \ll 1$, cannot reasonably go together, in general.

In the present paper we will examine further this issue raised by [1]. We argue that when compositeness is implemented in a consistent way, the Coleman–Weinberg phenomenon does not necessarily arise and fine-tuning may still be possible. This conclusion hinges in part upon the use of the perturbative renormalization group only in a regime in which it is valid. Indeed, in top condensation a la ref. [4], pains were taken to carefully match the low energy theory in which the perturbative coupling constant expansion is valid onto a high energy theory which approaches the scale Λ with a valid nonperturbative dynamics. We match a large- N_c expansion near Λ onto a perturbative renormalization group at some scale μ_i . Typically we choose

$\mu_i/\Lambda \sim 0.05$, but the results are reasonably insensitive to this choice.

We will reexamine the $U_L(N_f) \times U_R(N_f)$ chiral model of ref. [1] with elementary fermions having N_f flavors and N_c gauge degrees of freedom. Near the composite scale, Λ , the couplings become large and nonperturbative methods must be used to analyze the dynamics. We use the formal large- N_c methods as a guide to determine the renormalized coupling constants near the composite scale. At lower scales $\mu \ll \Lambda$, the effective couplings become weak and the usual perturbative methods become applicable. The large- N_c couplings are then matched at a scale μ_i onto the perturbative couplings which satisfy a two-loop renormalization group. Our criterion is that the two loop terms are no larger than the one-loop terms at μ_i , and this implies typically $\mu_i \sim 0.05$. We then evolve to the far infrared. We search for a Coleman-Weinberg instability at a scale v which may be intermediate to Λ and the far infrared scale of the low energy physics.

We will find using this procedure that these theories can admit, in general, large chiral hierarchies. In particular it turns out that taking into account two-loop corrections significantly stabilizes the $U_L(N_f) \times U_R(N_f)$ models, relative to the one-loop results. For example, the special case of $N_f = 2$, $N_c=5$, admits a chiral hierarchy larger than m_W/m_{pl} . Furthermore, in the presence of a strong gauge coupling like α_{QCD} we do not find any instability at all. We will study the stability of these conclusions and map out a class of models in which the chiral hierarchies range from $v/\Lambda \sim 10^{-2}$ to an arbitrarily infinitesimal v/Λ . It should be emphasized that all these results are physically somewhat qualitative, and not of the form of rigorous lemmas. However, once defined precisely, our procedure leads to rigorous results.

Of course, this does not have a bearing so far as we know on the origin of gauge hierarchies in nature. Certainly we would welcome a *raison d'être* for these hierarchies, such as, "if it can exist it must exist" (see e.g. [15]). The compositeness picture at the scale Λ suggests only that hierarchies are associated with the very close proximity of the high energy coupling parameters to the phase transition boundary. The approximate recovery of scale invariance in the evolution to low energies is associ-

ated with the tuning of the hierarchy and, in turn, with the infrared renormalization group fixed points [16, 17] that accompany the hierarchy. Perhaps at some deeper level these ingredients alone can be seen to self-consistently determine the existence of hierarchies.

II. The $U_L(N_f) \times U_R(N_f)$ Higgs–Yukawa model

In order to illustrate these ideas we take up the model considered in [1] as a generalization of top quark condensate models. It consists of N_f left- and right-handed fermion flavors Ψ^j ($j = 1, \dots, N_f$) which transform in an N_c -dimensional representation of some gauge group and possess a chiral $U_L(N_f) \times U_R(N_f)$ symmetry. The high-energy dynamics is assumed to produce a condensate at the scale Λ described by the (local) composite color singlet field $\Sigma^j \sim \bar{\Psi}_R^j \Psi_L^j$, whose VEV v may be interpreted as the order parameter of chiral symmetry breaking. It transforms in the $(\bar{\mathbf{N}}_f, \mathbf{N}_f)$ representation of the chiral symmetry group. If a large hierarchy between v and the scale Λ of new physics is to be established in a consistent way, the chiral symmetry breaking phase transition must be of second order in the couplings of the high-energy theory [19]. Hence the low-energy dynamics of the composite Higgs field Σ and the fermions Ψ^j can be described by an effective Ginsburg–Landau Lagrange density of the form

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + \frac{\pi y}{\sqrt{N_f}} (\bar{\Psi}_L \Sigma \Psi_R + \text{h.c.}) - V(\Sigma, \Sigma^\dagger) \quad (1)$$

where

$$V(\Sigma, \Sigma^\dagger) = m^2 \text{tr}(\Sigma^\dagger \Sigma) + \frac{\pi^2 \lambda_1}{3N_f^2} (\text{tr} \Sigma^\dagger \Sigma)^2 + \frac{\pi^2 \lambda_2}{3N_f} \text{tr}(\Sigma^\dagger \Sigma)^2 \quad (2)$$

is the most general renormalizable² classical potential symmetric under $U_L(N_f) \times U_R(N_f)$.

As already mentioned, this description of the effective low-energy dynamics presupposes a chiral symmetry breaking phase transition of *second order* in the coupling constants of the high-energy theory. However, as has been pointed out in [1], this can only be the case in a self-consistent way, if the phase transition in the effective low-energy theory (1), (2) is itself of second order. This means that its effective potential in the limit of vanishing renormalized mass should be minimized globally at $\Sigma_c = 0$, where Σ_c denotes the *classical* scalar field matrix. Any non-trivial global minimum at $\Sigma_c \neq 0$ for $m^2 = 0$, i.e. the occurrence of the Coleman-Weinberg phenomenon [14], would clearly indicate a first order transition of the low-energy theory, therefore rendering the crucial assumption of a second order transition of the high-energy theory self-inconsistent. This in turn would imply that a classically fine-tuned hierarchy v/Λ would always be destabilized by quantum fluctuations. We will investigate this potential problem following the approach of Yamagishi [18].

The appropriate technical tool to study the Coleman-Weinberg phenomenon is the effective potential. Its perturbative evolution to one-loop for the model (1), (2) is given by³

$$V_{\text{eff}} = V + \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln \left(1 + \frac{W}{p^2} \right) - \int \frac{d^4 p}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\pi y}{\sqrt{N_f} p^2} \right)^n \text{tr} \left[p_\mu \gamma^\mu (\Sigma_c P_R + \Sigma_c^\dagger P_L) \right]^n + \text{CT}, \quad (3)$$

where V is the tree-level potential (2) and $W \equiv W(\lambda)$ denotes the matrix of second derivatives of V w.r.t. the (real) scalar field components in Σ . CT is the counter

²For $N_f = 4$ there exists an additional independent renormalizable $U_L(N_f) \times U_R(N_f)$ -symmetric term of the form $\lambda_3 (\det \Sigma + \det \Sigma^\dagger)$. If the dynamics at $\mu = \Lambda$ is a gauge theory then presumably instantons can lead to the occurrence of such t'Hooft terms in the low-energy theory. We will ignore this possibility for simplicity.

³At this stage we neglect any possible gauge coupling of the fermions.

term for this UV-divergent expression and $P_{R/L} \equiv (1 \pm \gamma_5)/2$. Using dimensional regularization and \overline{MS} -renormalization this yields

$$\begin{aligned} V_{\text{eff}} = & V - \frac{\pi^2 N_c}{16 N_f^2} y^4 \left(\ln \left(\frac{\pi^2 y^2}{N_f} \right) - 2u - \frac{3}{2} \right) \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 \\ & - \frac{\pi^2 N_c}{16 N_f^2} y^4 \text{tr} \left[(\Sigma_c^\dagger \Sigma_c)^2 \ln \left(\frac{N_f \Sigma_c^\dagger \Sigma_c}{\text{tr} \Sigma_c^\dagger \Sigma_c} \right) \right] \\ & + \frac{1}{64 \pi^2} \text{tr} \left[W^2 \left(\ln \left(\frac{N_f W}{\text{tr} \Sigma_c^\dagger \Sigma_c} \right) - 2u - \frac{3}{2} \right) \right] \end{aligned} \quad (4)$$

with the scale parameter u defined as

$$u = \frac{1}{2} \ln \left(\frac{\text{tr} \Sigma_c^\dagger \Sigma_c}{N_f M^2} \right) . \quad (5)$$

and M being the renormalization scale.

The renormalization group equation for this potential may be written as

$$\mathcal{D} V_{\text{eff}}(\Sigma_c, \Sigma_c^\dagger) = 0 \quad (6)$$

with

$$\mathcal{D} = -\frac{\partial}{\partial u} + \overline{\beta}_1 \frac{\partial}{\partial \lambda_1} + \overline{\beta}_2 \frac{\partial}{\partial \lambda_2} + \overline{\beta}_y \frac{\partial}{\partial y} - \overline{\gamma} \sum_{i,j} \left(\Sigma_c^{ij} \frac{\partial}{\partial \Sigma_c^{ij}} + \Sigma_c^{\dagger ij} \frac{\partial}{\partial \Sigma_c^{\dagger ij}} \right) . \quad (7)$$

$$\overline{\beta}_1 = \frac{\beta_1}{1+\gamma} , \quad \overline{\beta}_2 = \frac{\beta_2}{1+\gamma} , \quad \overline{\beta}_y = \frac{\beta_y}{1+\gamma} , \quad \overline{\gamma} = \frac{\gamma}{1+\gamma} , \quad (8)$$

and the β 's denote the conventional beta functions

$$\beta_1 = \mu \frac{d \lambda_1}{d \mu} , \quad \beta_2 = \mu \frac{d \lambda_2}{d \mu} , \quad \beta_y = \mu \frac{d y}{d \mu} . \quad (9)$$

The anomalous dimension of Σ is defined as $\gamma = \frac{1}{2} \mu (d \ln Z_\Sigma / d \mu)$, where Z_Σ is the scalar wave function renormalization constant.

In view of (4) the one-loop or next-to-leading log renormalization group improved effective potential is then given by

$$\begin{aligned}
V_{\text{eff}}^{\text{RG}} = & \left\{ \frac{\pi^2}{3N_f^2} \bar{\lambda}_1(u, \lambda) (\text{tr } \Sigma_c^\dagger \Sigma_c)^2 + \frac{\pi^2}{3N_f} \bar{\lambda}_2(u, \lambda) \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 \right. \\
& - \frac{\pi^2}{16} \frac{N_c}{N_f^2} \bar{y}(u, \lambda) \text{tr} \left[(\Sigma_c^\dagger \Sigma_c)^2 \left(\ln \left(\frac{\pi^2 \bar{y}^2(u, \lambda) \Sigma_c^\dagger \Sigma_c}{\text{tr } \Sigma_c^\dagger \Sigma_c} \right) - \frac{3}{2} \right) \right] \\
& + \frac{1}{64\pi^2} \text{tr} \left[W^2(\bar{\lambda}(u, \lambda)) \left(\ln \left(\frac{N_f W(\bar{\lambda}(u, \lambda))}{\text{tr } \Sigma_c^\dagger \Sigma_c} \right) - \frac{3}{2} \right) \right] \Big\} \\
& \times \exp \left\{ -4 \int_0^u ds \bar{\gamma}(\bar{\lambda}(s, \lambda)) \right\}.
\end{aligned} \tag{10}$$

The effective coupling constants $\bar{\lambda}_1(u, \lambda)$, $\bar{\lambda}_2(u, \lambda)$ and $\bar{y}(u, \lambda)$ are defined by the generically coupled set of differential equations (the argument λ represents the dependence on λ_1 , λ_2 and y)

$$\bar{\beta}_1(\bar{\lambda}_1, \bar{\lambda}_2, \bar{y}) = \frac{\partial \bar{\lambda}_1}{\partial u}, \quad \bar{\beta}_2(\bar{\lambda}_1, \bar{\lambda}_2, \bar{y}) = \frac{\partial \bar{\lambda}_2}{\partial u}, \quad \bar{\beta}_y(\bar{\lambda}_1, \bar{\lambda}_2, \bar{y}) = \frac{\partial \bar{y}}{\partial u} \tag{11}$$

with boundary conditions $\bar{\lambda}_1(0, \lambda) = \lambda_1$, $\bar{\lambda}_2(0, \lambda) = \lambda_2$ and $\bar{y}(0, \lambda) = y$.

V_{eff} develops a local minimum in Σ_c different from zero if for some u_0 the following two conditions are fulfilled [18]:

$$\begin{aligned}
\frac{\partial V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{ij}} \Big|_{u=u_0} &= \frac{\partial V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{\dagger ij}} \Big|_{u=u_0} = 0 \\
W(u_0) &\equiv \begin{pmatrix} \frac{\partial^2 V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{ij} \partial \Sigma_c^{kl}} & \frac{\partial^2 V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{ij} \partial \Sigma_c^{\dagger kl}} \\ \frac{\partial^2 V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{\dagger ij} \partial \Sigma_c^{kl}} & \frac{\partial^2 V_{\text{eff}}^{\text{RG}}}{\partial \Sigma_c^{\dagger ij} \partial \Sigma_c^{\dagger kl}} \end{pmatrix} \Big|_{u=u_0} > 0.
\end{aligned} \tag{12}$$

The first relation indicates a stationary point of $V_{\text{eff}}^{\text{RG}}$, whereas the second ensures that this stationary point is indeed a local minimum of the effective potential. If furthermore $V_{\text{eff}}^{\text{RG}}(u_0) < 0$ is satisfied, this minimum is not only a local but a global one.

Later we will analyze these conditions in perturbation theory. A consistent one-loop treatment requires the knowledge of the one-loop $\bar{\beta}$ -functions but only the leading log (or RG-improved tree-level) terms of (10). In this case the conditions (12) may be rewritten as

$$\begin{aligned} S(u_0) &\equiv \left[4(\bar{\lambda}_1 + \bar{\lambda}_2) + \beta_1(\bar{\lambda}) + \beta_2(\bar{\lambda}) \right]_{u=u_0} = 0 \\ P(u_0) &\equiv \left[4 + \beta_1(\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}_1} + \beta_2(\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}_2} + \beta_y \frac{\partial}{\partial \bar{y}} \right] (\beta_1(\bar{\lambda}) + \beta_2(\bar{\lambda}))_{u=u_0} > 0 \quad (13) \\ \bar{\lambda}_2(u_0, \lambda) &> 0. \end{aligned}$$

The condition $V_{\text{eff}}^{\text{RG}}(u_0) < 0$ yields

$$\bar{\lambda}_1(u_0, \lambda) + \bar{\lambda}_2(u_0, \lambda) < 0. \quad (14)$$

Any solution to the condition $S(u_0) = 0$ can be understood as a quartic instability of the effective potential as is evident in the classical limit, i.e. for vanishing beta-functions. In addition it allows for a nice geometrical interpretation in coupling constant space [18]: Since the effective potential (as well as its extrema) is independent of the renormalization scale M (this is just the meaning of the RGE (6)) we may fix it to any physically meaningful value $M \leq \Lambda$. The flow of $\bar{\lambda}_1$, $\bar{\lambda}_2$ and \bar{y} with $\mu^2 \equiv \text{tr}(\Sigma_c^\dagger \Sigma_c / N_f)$ then determines whether there is an additional extremum away from $\mu = 0$. If for some $u_0 = \ln(\mu_0/M)$ the flow intersects the "stability surface" $S(u) = 0$ and in addition the other conditions in (13) and (14) are satisfied there is an absolute minimum of V_{eff} away from zero and the VEV of Σ is given by

$$\frac{1}{\sqrt{2}} v_0 \delta^{ij} \equiv \langle 0 | \Sigma^{ij} | 0 \rangle = M \epsilon^{u_0} \delta^{ij}. \quad (15)$$

In a second step we will improve the one-loop results by including two-loop corrections. A consistent treatment then requires the two-loop corrections to the $\bar{\beta}$ -functions as well as the next-to-leading log contributions to the RG-improved effective potential as given in (10). Consequently the functions $S(u)$ and $P(u)$ acquire higher order corrections.

Eqn. (14) already shows that the phase transition of the minimal top condensation model [4], which has only one quartic coupling constant, is always of second order. The compositeness boundary conditions guarantee that its infrared stable quasi-fixed point is at $\lambda > 0$ and (14) can not be fulfilled. This is, however, not a general feature of models with only one quartic coupling constant, since e.g. scalar QED clearly exhibits the Coleman-Weinberg phenomenon [14]⁴ and is expected to have a first order phase transition.

III. An exactly solvable example

The Lagrange density (1), (2) may be viewed as the effective low-energy Landau-Ginsburg description of a gauged Nambu-Jona-Lasinio model [20]. This is a quantum theory with a cut-off at the scale Λ , and should generally be viewed as an approximation to a more general Lagrangian involving a series of higher dimension operators:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + G \left(\bar{\Psi}_L^i \Psi_R^j \right) \left(\bar{\Psi}_R^j \Psi_L^i \right) \quad (16)$$

The equivalence is seen by rewriting (16) in a Yukawa form with the help of a static auxiliary scalar matrix field Σ (see e.g. [21]):

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + y_0 \left(\bar{\Psi}_L \Sigma \Psi_R + h.c. \right) - m_0^2 \text{tr} \left(\Sigma^\dagger \Sigma \right) \quad (17)$$

The physical low energy effective theory is generated by integrating out fermion degrees of freedom with momentum $\mu < p < \Lambda$. This effective theory cannot contain any physical implications for physics on scales $p > \Lambda$. At scales $\mu \ll \Lambda$ the Yukawa interaction in (17) induces the fully gauge invariant, kinetic and quartic scalar self-interaction terms of (2). In the fermion bubble approximation, i.e. to leading order in

⁴It should be noted that the third condition (13) is due to the fact that the model under consideration has two quartic scalar self-interactions. A similar condition does not show up in models with only one quartic coupling like scalar QED.

a large- N_c expansion, GN_c fixed and all gauge couplings neglected, this model is exactly solvable. In this approximation the beta functions for the renormalized coupling constants and the anomalous dimension of the scalar field are given by:

$$\beta_1^{(1/N_c)} = 0, \quad \beta_2^{(1/N_c)} = 4a\lambda_2 y^2 - 6ay^4, \quad \beta_{y^2}^{(1/N_c)} = 2ay^4 \quad (18)$$

$$\gamma^{(1/N_c)} = ay^2 \quad (19)$$

with $a \equiv N_c/16N_f$. The corresponding renormalized couplings are seen to be [4]:

$$\lambda_1(\mu) = 0, \quad \lambda_2(\mu) = \frac{24N_f}{N_c \ln(\frac{\Lambda}{\mu})}, \quad y^2(\mu) = \frac{8N_f}{N_c \ln(\frac{\Lambda}{\mu})}, \quad (20)$$

i.e. λ_2 and y^2 tend to diverge at the compositeness scale. We will always assume in the following that $N_c \geq N_f$. Otherwise the large- N_c expansion would no longer be reliable and instead would have to be replaced by a large- N_f expansion yielding presumably very different infrared dynamics.

For a further discussion of the nature of the chiral symmetry breaking we have to determine the flow of the coupling constants $\bar{\lambda}_1$, $\bar{\lambda}_2$ and \bar{y}^2 which is governed by the beta functions defined in (8). The solution to (11) can only be given in a transcendental form:

$$\bar{\lambda}_1(u) = 0, \quad \bar{\lambda}_2(u) = 3\bar{y}^2(u), \quad 2u = \ln \left(\frac{\bar{y}^2(u)}{y^2} \right) - \frac{1}{a} \left(\frac{1}{\bar{y}^2(u)} - \frac{1}{y^2} \right). \quad (21)$$

The full quantum corrections to the effective potential in this approximation are given by the fermionic one-loop contributions in (4):

$$\begin{aligned} V_{\text{eff}} &= \frac{\pi^2 \lambda_2}{3N_f} \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 - \frac{\pi^2 N_c}{16 N_f^2} y^4 \left(\ln \left(\frac{\pi^2 y^2}{N_f} \right) - \frac{3}{2} \right) \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 \\ &- \frac{\pi^2 N_c}{16 N_f^2} y^4 \text{tr} \left[(\Sigma_c^\dagger \Sigma_c)^2 \ln \left(\frac{N_f \Sigma_c^\dagger \Sigma_c}{M^2} \right) \right]. \end{aligned} \quad (22)$$

Since this potential is exact to leading order in N_c it can not be improved by the RG (to this order). Hence it should be possible to rewrite it in a manifestly RG-invariant

way. Indeed, absorbing its u -dependence into the effective coupling constants using (21) yields the desired form of the effective potential:

$$\begin{aligned}
V_{\text{eff}} = & \left\{ \frac{\pi^2 \bar{\lambda}_2(u)}{3N_f} \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 - \frac{\pi^2 N_c}{16 N_f^2} \bar{y}^4(u) \left(\ln \left(\frac{\pi^2 \bar{y}^2(u)}{N_f} \right) - \frac{3}{2} \right) \text{tr} (\Sigma_c^\dagger \Sigma_c)^2 \right. \\
& \left. - \frac{\pi^2 N_c}{16 N_f^2} \bar{y}^4(u) \text{tr} \left[(\Sigma_c^\dagger \Sigma_c)^2 \ln \left(\frac{N_f \Sigma_c^\dagger \Sigma_c}{\text{tr} \Sigma_c^\dagger \Sigma_c} \right) \right] \right\} \\
& \times \exp \left\{ -4 \int_0^u ds \bar{\gamma}(s) \right\}
\end{aligned} \tag{23}$$

which exactly agrees with the fermionic contribution to (10).

The stability of this potential may be investigated by recasting (22) with the help of (20) to become

$$V_{\text{eff}} = -\frac{\pi^2}{N_f} a y^4 \text{tr} \left[(\Sigma_c^\dagger \Sigma_c)^2 \ln \left(\frac{\pi^2 y^2 \Sigma_c^\dagger \Sigma_c}{N_f \Lambda^2 e^{3/2}} \right) \right]. \tag{24}$$

One sees that the large- N_c fermionic contribution to the effective potential is stable for allowed values of the classical field, $(\text{tr} \Sigma_c^\dagger \Sigma_c) < N_f \Lambda^2$. For larger values of the classical field, the expression for the effective potential in (24) is not valid, the fermions actually decouple and we can not integrate down to the infrared scale. The apparent instability of (24) for large values of the classical field is completely unphysical. The critical issue and result is that we see no instabilities corresponding to intermediate scales $\mu < \langle \Sigma \rangle < \Lambda$. Thus, the only relevant physical local minimum remains at the origin, and the theory is consistently stable against the intermediate Coleman-Weinberg instability in the large- N_c limit.

IV. The perturbative regime

Though the fermion bubble approximation provides a nice method to solve the model described by (16) in the strong coupling regime, i.e. close to the scale Λ , it is a

crude approximation for small N_c , which might give at most some qualitative physical hints, for the nonperturbative regime $\mu > \mu_i$ and especially the values of the coupling constants at $\mu = \mu_i$ once the full theory is considered. Below μ_i the perturbative expansion of the beta functions provides a much more accurate description of how the couplings run with scale. However, we emphasize that reliable physical information may not necessarily be obtained by using only the lowest order terms of the perturbative beta functions in the renormalization group equations as done in [1]. At scales near Λ the couplings are becoming large and higher order terms may be essential to determine the evolution. We will demonstrate this below numerically by comparing the one-loop running of the couplings to the two-loop running at high scales. At higher scales, the perturbative renormalization group fails and the couplings must be matched to the nonperturbative dynamics near the composite scale such as provided by the large- N_c running of $\bar{\lambda}_1$, $\bar{\lambda}_2$ and \bar{y}^2 to that governed by the perturbative beta functions at the scale μ_i . Our next task will therefore be the determination of μ_i or equivalently the perturbative regime of coupling constants.

A simple way to obtain a first idea about the regime of couplings where perturbation theory can be trusted is to derive tree-level unitarity bounds in the approximation of vanishing Yukawa coupling y^2 . The strongest restrictions are found by considering $\Sigma^\dagger \Sigma$ scattering. One obtains the unitarity bounds

$$\begin{aligned} \left| \frac{1}{2} \left(1 + \frac{1}{N_f^2} \right) \lambda_1 + \lambda_2 \right| &\leq \frac{6}{\pi} \\ \left| \frac{1}{N_f^2} \lambda_1 \right| &\leq \frac{12}{\pi} \\ \left| \frac{1}{N_f^2} \lambda_1 + \lambda_2 \right| &\leq \frac{12}{\pi} \end{aligned} \tag{25}$$

for the singlet-singlet, adjoint-adjoint and singlet-adjoint channels, respectively⁵. For the large- N_c boundary condition $\lambda_1 = 0$ this yields in particular $\lambda_2 \leq 1.9$.

⁵Note that $(\mathbf{N}_f, \bar{\mathbf{N}}_f) \otimes (\bar{\mathbf{N}}_f, \mathbf{N}_f) = (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{N}_f^2 - 1) \oplus (\mathbf{N}_f^2 - 1, \mathbf{1}) \oplus (\mathbf{N}_f^2 - 1, \mathbf{N}_f^2 - 1)$.

Another possibility to determine the perturbative regime in coupling constant space is to compare the magnitude of the full $n + 1$ -loop contribution to some perturbative quantity to its full n -loop contribution. For our purpose the most natural choice are the beta functions defined in (8). The coupling constants are certainly outside the perturbative regime, if the two-loop corrections to their beta functions are bigger than the one-loop results. Furthermore, since we are interested in using the perturbative evolution of the coupling constants as far as possible, i.e. limiting the crude large- N_c approximation to the smallest possible range of the scale parameter, the determination of the full two-loop corrections to the beta functions is mandatory in order to minimize perturbative errors.

The one-loop contributions are given by

$$\begin{aligned}
 \beta_1^{(1)} &= \mu \frac{d\lambda_1}{d\mu} &= \frac{1}{3}\lambda_1^2 \left(\frac{1}{4} + \frac{1}{N_f^2} \right) + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{4}\lambda_2^2 + \frac{1}{4}\frac{N_c}{N_f}\lambda_1 y^2 \\
 \beta_2^{(1)} &= \mu \frac{d\lambda_2}{d\mu} &= \frac{1}{6}\lambda_2^2 + \frac{1}{2N_f^2}\lambda_1\lambda_2 + \frac{1}{4}\frac{N_c}{N_f}\lambda_2 y^2 - \frac{3}{8}\frac{N_c}{N_f}y^4 \\
 \beta_{y^2}^{(1)} &= \mu \frac{dy^2}{d\mu} &= \frac{1}{8} \left(1 + \frac{N_c}{N_f} \right) y^4 \\
 \gamma^{(1)} &= \frac{1}{16} \frac{N_c}{N_f} y^2 .
 \end{aligned} \tag{26}$$

Since $\gamma^{(1)} \sim O(y^2)$ we have $\bar{\beta}_j^{(1)} = \beta_j^{(1)}$ ($j = 1, 2, y^2$).

The two-loop corrections may be evaluated using the results of [22]. One obtains

$$\begin{aligned}
\beta_1^{(2)} &= -\frac{1}{24} \left[\lambda_2^3 + \frac{1}{12} \left(5 + \frac{41}{N_f^2} \right) \lambda_2^2 \lambda_1 + \frac{11}{3N_f^2} \lambda_2 \lambda_1^2 + \frac{1}{4N_f^2} \left(3 + \frac{7}{N_f^2} \right) \lambda_1^3 \right] \\
&- \frac{1}{24} \frac{N_c}{N_f} \left[\frac{3}{4} \lambda_2^2 + \lambda_2 \lambda_1 + \left(\frac{1}{4} + \frac{1}{N_f^2} \right) \lambda_1^2 \right] y^2 \\
&+ \frac{1}{32} \frac{N_c}{N_f} \left[\left(\lambda_2 - \frac{3}{4} \lambda_1 \right) y^4 + \frac{3}{2} y^6 \right] \\
\beta_2^{(2)} &= -\frac{1}{24} \left[\frac{1}{12N_f^2} \left(5 + \frac{41}{N_f^2} \right) \lambda_1^2 \lambda_2 + \frac{11}{3N_f^2} \lambda_1 \lambda_2^2 + \frac{1}{4} \left(1 + \frac{5}{N_f^2} \right) \lambda_2^3 \right] \\
&- \frac{1}{16} \frac{N_c}{N_f} \left[\left(\frac{1}{N_f^2} \lambda_1 \lambda_2 + \frac{1}{3} \lambda_2^2 \right) y^2 + \left(\frac{3}{8} \lambda_2 - \frac{1}{2N_f^2} \lambda_1 \right) y^4 - \frac{3}{4} y^6 \right] \\
\beta_{y^2}^{(2)} &= \frac{1}{576} \left[\frac{1}{N_f^2} \left(1 + \frac{1}{N_f^2} \right) \lambda_1^2 + \frac{4}{N_f^2} \lambda_1 \lambda_2 + \left(1 + \frac{1}{N_f^2} \right) \lambda_2^2 \right] y^2 \\
&- \frac{1}{48} \left[\frac{1}{N_f^2} \lambda_1 + \frac{1}{2} \left(1 + \frac{1}{N_f^2} \right) \lambda_2 \right] y^4 - \frac{1}{64} \left[\frac{1}{8} + \frac{3}{2} \frac{N_c}{N_f} - \frac{1}{N_f^2} \right] y^6.
\end{aligned} \tag{27}$$

For simplicity we have neglected the effects of gauge couplings in these expressions. This may only be justified at high scales where they may be assumed to be small. We will comment later on the modifications induced by a sizable gauge coupling, e.g. α_{QCD} , on the running of \bar{y}^2 , $\bar{\lambda}_1$ and $\bar{\lambda}_2$.

Since $\gamma^{(2)}$ is of the same order in the coupling constants as the two-loop contributions to the beta functions we find at the two-loop level

$$\bar{\beta}_j = \left[1 - \gamma^{(1)} \right] \beta_j^{(1)} + \beta_j^{(2)}; \quad j = 1, 2, y^2. \tag{28}$$

Hence the evolution of $\bar{\lambda}_1$ close to the compositeness boundary conditions ($\bar{\lambda}_1 = 0$, $\bar{y}^2 = \bar{\lambda}_2/3$) to two loops is dominated by

$$\bar{\beta}_1|_{\bar{\lambda}_1=0, \bar{y}^2=\bar{\lambda}_2/3} = \frac{1}{4} \bar{\lambda}_2^2 - \frac{1}{24} \left[1 + \frac{1}{4} \frac{N_c}{N_f} \right] \bar{\lambda}_2^3. \tag{29}$$

The positive one-loop contribution to $\bar{\beta}_1$ drives $\bar{\lambda}_1$ negative as one evolves downwards from μ_i . This is qualitatively changed if the absolute value of the (negative) two-loop contribution becomes larger than the one-loop result. $\bar{\lambda}_1$ is then driven into the positive region as displayed in Fig. 1 and for even larger initial values of $\bar{\lambda}_2$ both scalar self-interaction couplings develop an infrared singularity. Fig. 1 furthermore shows that the low-energy physics of this model is fairly insensitive to the exact initial values especially if they are large and the two-loop evolution of the coupling constants is used. This fact is due to the existence of an effective infrared fixed point in coupling constant space [16, 17].

In view of (29) we therefore conclude that the perturbative running of the coupling constants with large- N_c initial values is definitely misleading if $\bar{\lambda}_2$ is larger than

$$\bar{\lambda}_{2i}^{\text{pert}} \equiv \frac{24N_f}{4N_f + N_c} \quad (30)$$

since then the perturbative error is about 100%. In fact, in order to avoid large perturbative errors especially in the high t -regime ($t = \ln(\Lambda/\mu)$), we have to choose a starting value $\bar{\lambda}_{2i} \equiv \bar{\lambda}_2(\mu_i)$ which is somewhat smaller than $\bar{\lambda}_{2i}^{\text{pert}}$. On the other hand, one should not use the large- N_c running of the couplings for too many orders of magnitudes in the scale parameter. In order to obtain the smallest possible numerical errors in our calculation we will therefore match the perturbative to the large- N_c running of the coupling constants at a scale μ_i for which

$$\frac{1}{2}\bar{\lambda}_{2i}^{\text{pert}} < \bar{\lambda}_{2i} < \bar{\lambda}_{2i}^{\text{pert}} \quad (31)$$

or equivalently

$$1 + 4\frac{N_f}{N_c} < \ln\left(\frac{\Lambda}{\mu_i}\right) < 2\left(1 + 4\frac{N_f}{N_c}\right) \quad (32)$$

This means that for e.g. $N_f = 2$, $N_c = 3$ (5) one has to use the $1/N_c$ -running for approximately 1.6 (1.1) orders of magnitude, in order to reach the matching scale μ_i . If one prefers however, to chose a matching scale μ_i at which the perturbative error

is reduced to about 50% as for the lower bound of (31) one has to use the large- N_c running for 3.2 (2.2) orders of magnitude.

One might feel somewhat uneasy about using the bubble approximation over such a large range in the scale parameter especially for small N_c . We have paid regard to this concern in our numerical analysis by varying the initial values of the coupling constants considerably around their large- N_c values at μ_i . On the other hand it is also known that higher dimensional operators in the NJL-Lagrangian, which can be determined once the precise high-energy dynamics above the scale Λ is known, can also change the compositeness conditions [23]. $\bar{\lambda}_2$ generically takes on a large but *finite* value at Λ once these operators are taken into account. This will result in a somewhat shorter and perhaps even faster evolution down to $\bar{\lambda}_{2i}^{\text{pert}}$. Furthermore we expect subleading corrections to the large N_c -running close to Λ to have similar effects.

V. Numerical results

For physical applications it is important to establish the range of hierarchies permitted by a given model. Not only is it essential to determine if the evolution of the coupling constants intersects the stability surface, but it is also to establish how many orders of magnitude one can run before this happens, i.e. how large a hierarchy v/Λ one can establish without running into a Coleman-Weinberg instability. In addition one would like to know how this depends on N_f and N_c . Finally one should investigate how sensitive the evolution and its stability are w.r.t. a modification in the running of \bar{y}^2 , e.g. due to the influence of a fairly strong gauge interaction like $SU(3)_{\text{QCD}}$. We will address these issues in the following.

To get a first idea of how the coupling constants evolve with scale one can (as done in [1]) make the simplifying assumption of a constant Yukawa coupling, i.e. $\bar{y}^2 = 1$. In this case the stability hyper-surface is reduced to a line. In Fig. 1 we have plotted the one- and two-loop perturbative flow in coupling constant space for this situation

for various large- N_c initial values and $N_f = 2$, $N_c = 3$ including this stability line. The perturbative results are trustworthy up to an initial value of $\bar{\lambda}_{2i}^{\text{pert}} \approx 4.6$ for this case. Our numerical results agree with those of [1], though, as explained before, our interpretation is different. The instabilities found in [1] all correspond to initial values of the coupling constants for which perturbation theory breaks down. We therefore choose to apply the large- N_c approximation to model the dynamics in the strong-coupling regime until perturbative values are reached. Taking these as initial values for the perturbative evolution the RG-trajectories *never* cross the stability line therefore indicating a phase transition of *second* order. For larger initial values the two-loop evolution clearly shows that the perturbative results can no longer be trusted. We therefore conclude that compositeness boundary conditions may well be compatible with a second order phase transition. Furthermore one notices from Fig. 1 that the two-loop evolution greatly improves the stability of the model over the one-loop running. This may however be a model-dependent result. Another feature of the numerical analysis is that the stability increases with increasing N_c for fixed N_f . This is, however, to be expected from our large- N_c analysis of section III.

A constant \bar{y}^2 does of course not correspond to a physically motivated model. The simplest assumption corresponding to a real model is that of a vanishing Yukawa coupling. In this case our methods clearly signal a *first* order phase transition independent of the values of N_f and N_c for a wide range of perturbative initial values for the coupling constants. This agrees with the results of [24] and [25] and is not too surprising because of the lack of infrared fixed points in coupling constant space. For our purpose, however, this situation is only of little interest, since a vanishing Yukawa coupling is not consistent with the compositeness conditions we wish to investigate.

We will therefore turn to the full perturbative running of \bar{y}^2 as given by (26) and (27). The flow of the coupling constants resembles that of Fig. 1. In Figs. 2 and 3 we have plotted the perturbative evolution of the stability function $S(u)$ (as defined in (13) for a leading log analysis) for the cases $N_f = 2$, $N_c = 3$ and $N_f = 2$, $N_c = 5$, respectively, for the leading as well as for the next-to-leading log expression of the effective potential and for various large- N_c motivated initial values of the couplings

inside the perturbative regime. In the case of $N_c = 3$ the function S generally exhibits a zero which can be seen to correspond to a local minimum of $V_{\text{eff}}^{\text{RG}}$ indicating a first order phase transition. However, one should notice that the zeros occur after a significant amount of running therefore allowing for the tuning of hierarchies of approximately $\Lambda/v \approx 10^{10}$ if the two-loop evolution is used. Again we observe that the NLL-corrections as well as an increasing N_c seem to stabilize the potential for this particular model considerably.

Though the situation of a constant \bar{y}^2 considered earlier is not fully realistic, it mimics the low-scale behavior of the Yukawa coupling in the presence of a strong asymptotically free gauge coupling. Such a coupling would significantly affect the running of \bar{y}^2 at lower scales leading to an effective infrared fixed point [16, 17] which in turn might suggest, that the phase transition is of second order. This is of course also a more realistic configuration for physical applications in heavy quark condensation or strong ETC models.

For definiteness we will assume in the following that the gauge group is $SU(N_c)$ and that the fermions transform in the fundamental representation. The beta function for \bar{y}^2 at the one-loop level is then given by

$$\begin{aligned}\beta_{y^2}^{(1)} &= \frac{1}{8} \left(1 + \frac{N_c}{N_f} \right) y^4 - \frac{3(N_c^2 - 1)}{2\pi N_c} \alpha_s y^2 \\ \beta_{\alpha_s}^{(1)} &= -\frac{1}{\pi} \left(\frac{11}{6} N_c - \frac{1}{3} N_f \right) \alpha_s^2\end{aligned}\tag{33}$$

with $\alpha_s = g^2/4\pi$. The effect of the additional term for β_{y^2} is certainly small at high scales due to asymptotic freedom (for not too large N_f). In the infrared however, it becomes important and effectively stops the running of y^2 due to its negative sign. This situation is hence somewhere in between the running of \bar{y}^2 according to (26) and a constant Yukawa coupling. We therefore expect a somewhat more stable evolution of $\bar{\lambda}_1$ and $\bar{\lambda}_2$.

In Figs. 4 and 5 we have plotted the function $S(t)$ for $N_f = 2$, $N_c = 3$ and $N_f = 2$, $N_c = 5$, respectively, using various large- N_c motivated initial values $\bar{\lambda}_{2i} \leq \bar{\lambda}_{2i}^{\text{pert}}$.

We have normalized α to be approximately of the size of the QCD-coupling, i.e. $\alpha_s(m_Z) = 0.1$. Because of its smallness, especially at high scales, we have included only corrections of order α in the beta functions, i.e. the additional term in (33). For further definiteness we have chosen $\Lambda = m_{\text{Pl}} \approx 10^{19} \text{GeV}$.

Figs. 4 and 5 show that the two-loop evolution of the couplings never crosses the stability surface $S(t) = 0$. Hence one can self-consistently fine-tune a hierarchy between the weak and even the Planck scale without running into a Coleman-Weinberg instability. Remarkably, at comparably high initial values for $\bar{\lambda}_{2i}$ for which the one-loop evolution signals a potential instability, the situation is improved considerably by the NLL corrections. We have checked that these statements are insensitive to changes in N_f and N_c as long as N_f is not much larger than N_c . In particular we find, as expected by our large- N_c analysis and demonstrated in Fig. 5, that the stability increases again with increasing N_c for N_f held fixed.

Finally we should mention that we have checked the sensitivity of our numerical analysis w.r.t. small deviations from the large- N_c initial values for $\bar{\lambda}_1$ and \bar{y}^2 according to a parameterization

$$\bar{\lambda}_{1i} = a, \quad \bar{y}_i^2 = \frac{1}{3} \bar{\lambda}_{2i} [1 + b]. \quad (34)$$

We find that for reasonably small values of a and b , i.e. $|a|, |b| \leq 0.3$, our quantitative results are fairly insensitive therefore not altering our qualitative statements.

The establishment of large hierarchies by fine-tuning can be very model dependent as emphasized by the authors of [1]. Specific models may require careful analysis to establish the range of hierarchies which may be achieved by the dynamics. Even models with several effective couplings at low energy may be able to support large hierarchies as in the examples studied in this paper.

VI. Conclusions

In this paper we have argued that fine-tuned chiral hierarchies and compositeness boundary conditions on coupling constants can go together in a self-consistent way, avoiding the general problem of intermediate Coleman-Weinberg instabilities. Our analysis has focused on a chiral $U_L(N_f) \times U_R(N_f)$ -symmetric Yukawa-Higgs-model, though the methods applied are more general. By calculating the renormalization group improved effective potential including all next-to-leading log contributions, and implementing the matching of the perturbative running to the large- N_c evolution, we are able to provide numerical evidence that fine-tuned hierarchies are not endangered by the Coleman-Weinberg instability. We carefully match the nonperturbative, large- N_c running of the coupling constants, as they become large when approaching the compositeness scale Λ , to their two-loop perturbative RG-evolution within a valid range of applicability. This allows us to obtain reliable results for the full range of momentum scales, up to the compositeness scale. Curiously, the phenomenologically most relevant case, that of a strong non-abelian gauge coupling to the fermions, allows fine-tuning hierarchies as large as m_W/m_{pl} , owing to the presence of an effective nontrivial infrared fixed point. We find generally that the next-to-leading log corrections improve the stability over the leading log or one-loop results. Furthermore, the stability of the hierarchy generally increases significantly with growing N_c .

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Figure 1: *Perturbative one- and two-loop renormalization group trajectories in $\bar{\lambda}_1$ - $\bar{\lambda}_2$ space for $N_f = 2$, $N_c = 3$ and $\bar{y}^2 = 1$. The evolution is displayed for a running of 18 orders of magnitude in μ for several large- N_c initial values. The arrows point in the direction of the flow with decreasing scale. Each dot in one of the curves indicates an evolution by one order of magnitude.*

Figure 2: *This Figure shows the one- and two-loop evolution of the stability function $S(t)$ with t without gauge coupling for $N_f = 2$, $N_c = 3$ and the full running of \bar{y}^2 .*

Figure 3: *The same as Figure 2 but with increased number of fermion colors $N_c = 5$.*

Figure 4: *One- and two-loop evolution of the stability function $S(t)$ with t for $N_f = 2$ and $N_c = 3$ and the running of \bar{y}^2 modified by a strong gauge coupling α_s .*

Figure 5: *The same as Figure 3 but with increased number of fermion colors $N_c = 5$.*

Figure 1

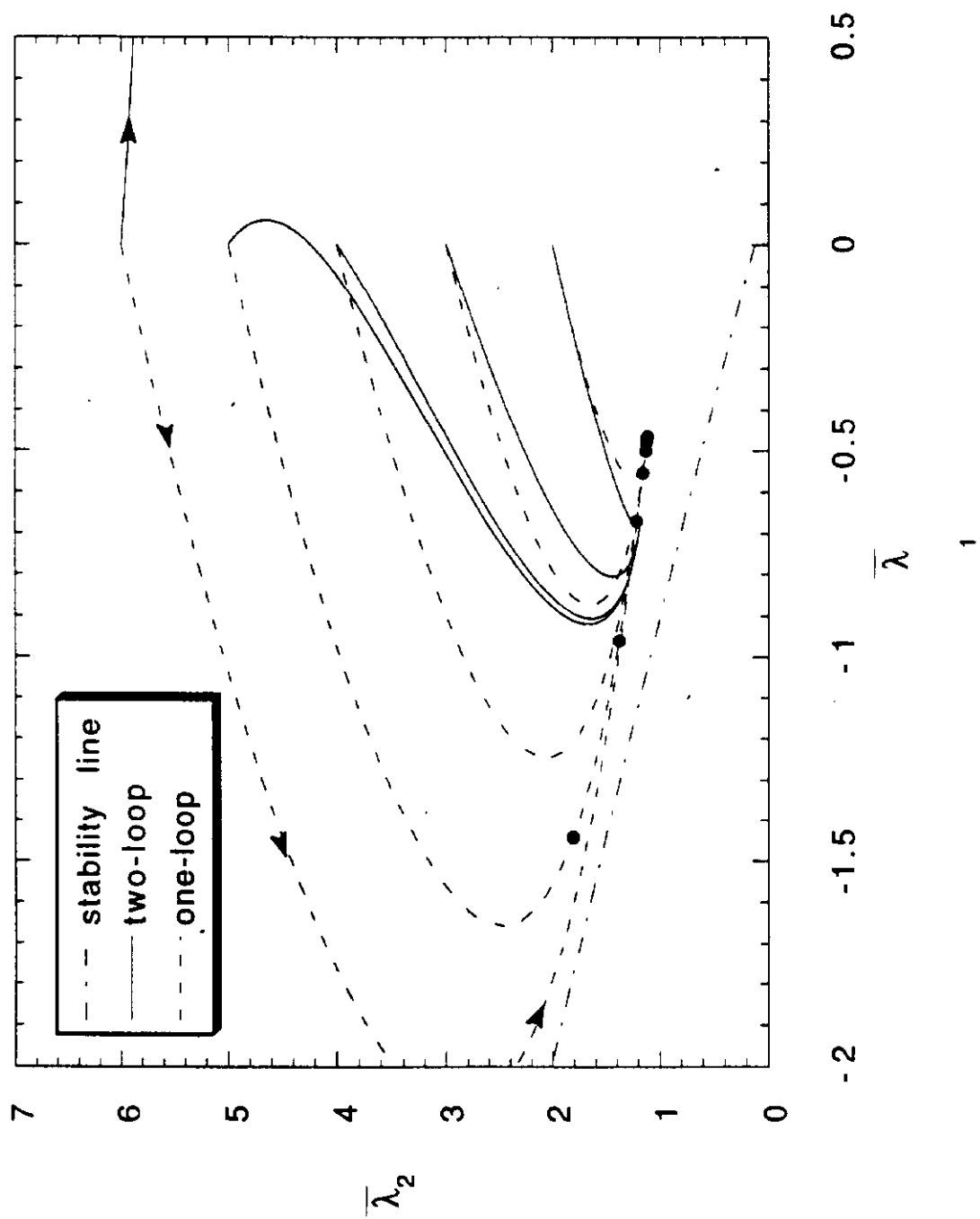


Figure 2

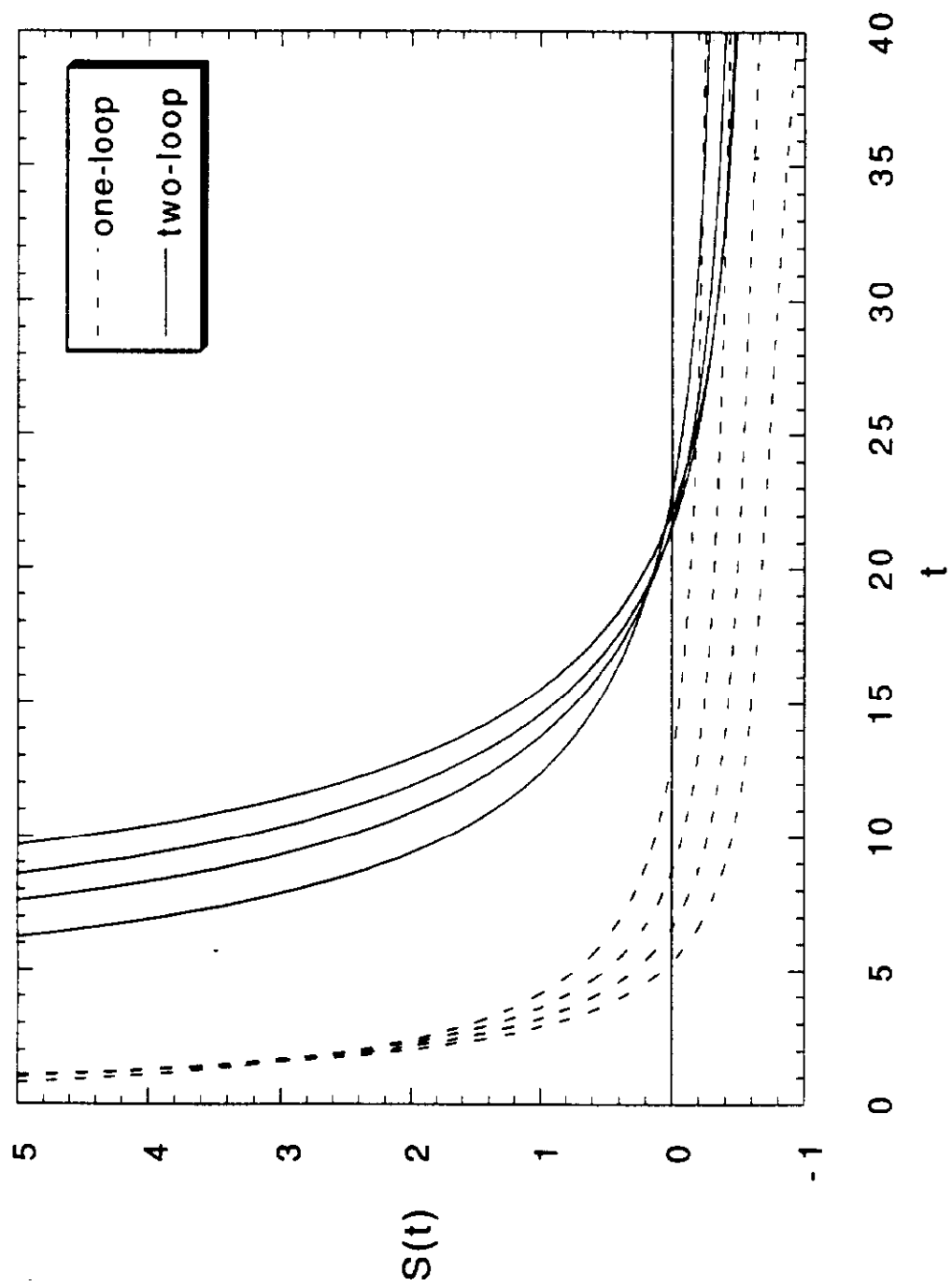


Figure 3

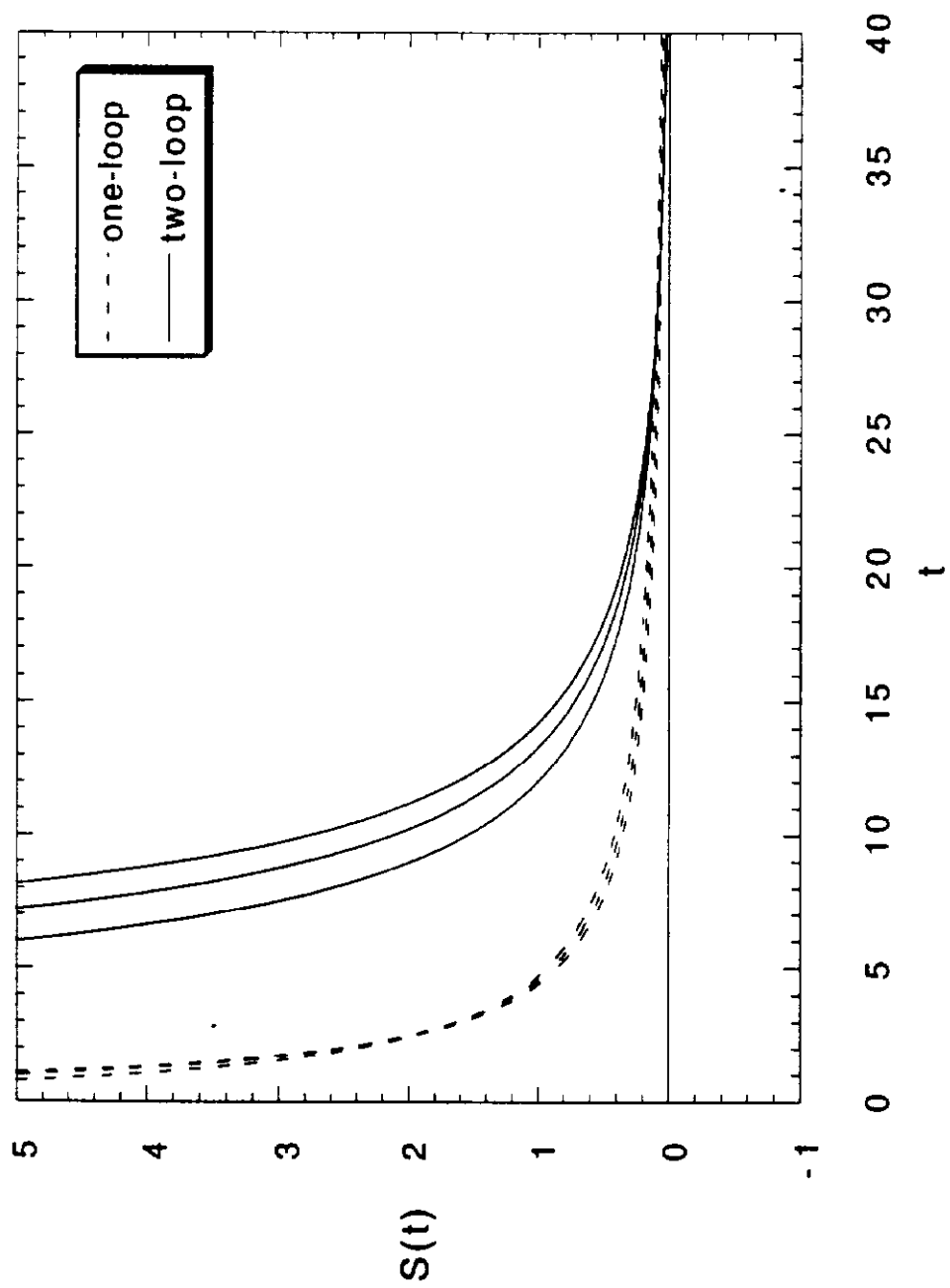


Figure 4

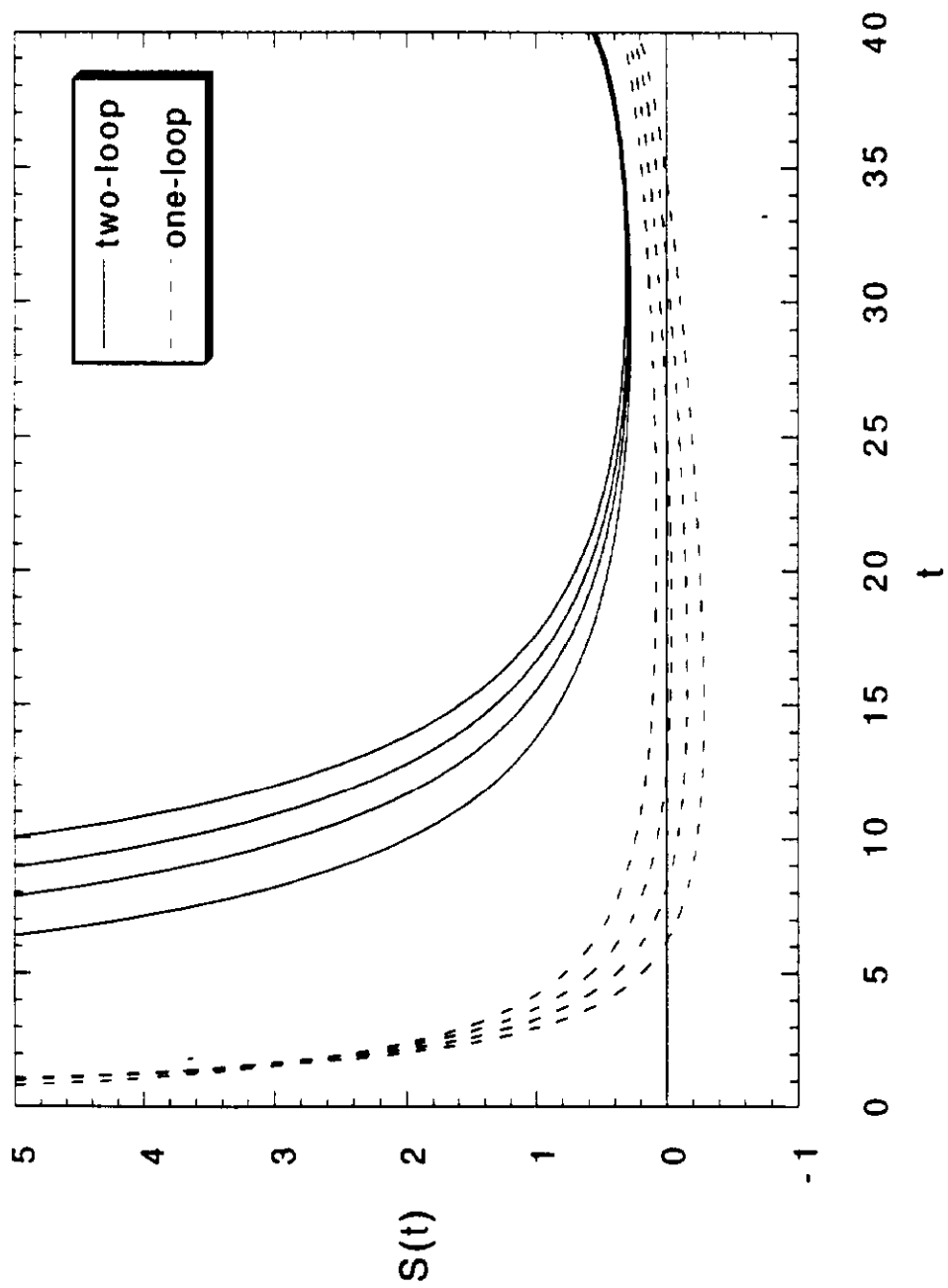


Figure 5

